CPT and Lorentz violation effects in hydrogen-like atoms *

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Within the framework of Lorentz-violating extended electrodynamics, the Dirac equation for a bound electron in an external electromagnetic field is considered assuming the interaction with a CPT-odd axial vector background b_μ. The quasi-relativistic Hamiltonian is obtained using a $1/c$-series expansion. Relativistic Dirac eigenstates in a spherically-symmetric potential are found accurate up to the second order in b_0. b_0-induced CPT-odd corrections to the electromagnetic dipole moment operators of a bound electron are calculated that contribute to the anapole moment of the atomic orbital and may cause a specific asymmetry of the angular distribution of the radiation of a hydrogen atom.

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I. INTRODUCTION

The Standard Model is currently proved with a convincingly wide set of experiments. Nevertheless, it essentially does not include a quantum description of gravitation. The quantization methods adopted in the Standard Model do not allow a self-consistent quantization of General Relativity since the subsequent theory occurs to be nonrenormalizable. Thus the essence of the theory accounting for the effects taking place at the *Planck scale* of energies ($E_{\rm Pl} = 10^{19}$GeV) where quantum gravity plays a major role, still remains obscure. At the same time, there exist some candidates for such Fundamental theory, string theory for instance, taking the form of the Standard Model in the low-energy limit.

Planck energies being far from experimental attainment, the *Standard Model Extension* (SME) was elaborated. It is an effective theory (applicable at the energies $E \ll E_{\rm Pl}$) formulated axiomatically as a set of corrections to the Lagrangian of the Standard Model fulfilling some 'natural' requirements [1, 2] such as observer Lorentz invariance, 4-momentum conservation, unitarity, and microcausality. In what follows, we will focus on a subset of the SME referred to as the *minimal* SME in *flat* Minkowsky spacetime that also requires local $SU(3)_C \times SU(2)_I \times U(1)_Y$ gauge invariance and power-counting renormalizability. A spectacular feature of such requirements is that they reduce the diversity of possible corrections down to a *finite* number of them. Each correction term consists of a complex (pseudo)tensor constant (SME coefficient) contracted with conventional Standard Model fields and their spacetime derivatives. These constants are believed to stand for vacuum expectation values of the fields featuring in the hypothetic Lorentz-covariant Fundamental theory and condensed at low energies due to the spontaneous symmetry breaking mechanism. Indeed, it has been shown recently that such Lorentz symmetry breaking can occur in some theories beyond the Standard Model [3, 4, 5, 6] leading subsequently to the SME. The SME can thus be used to reduce the complexity of these theories and related calculations in the low-energy limit. It also provides a standard for representation of data obtained in experiments searching for Lorentz violation.

Recently, a number of theoretical researches have been performed aiming at investigating the vacuum structure of this model (see, e.g. [7, 8, 9, 10, 11]), and to study the assumed violation on various high-energy processes [12, 13, 14, 15]. This search also seems quite promising in atomic physics [16, 17]. For instance, specific types of Lorentz violation may cause spatial parity violation in electrodynamics at tree level. P-parity violation effects in atomic systems *within the conventional Standard Model* have been thoroughly studied either theoretically or experimentally in the past four decades [18, 19, 20, 21, 22]. Such effects are caused by weak interaction and include resonant dichroism of atomic gas, permission of conventionally forbidden quantum transitions etc. Much the same effects are expected within the SME.

Until today however, studies of atom within the SME have included only spectroscopic predictions using the perturbation theory with respect to SME coefficients [16, 17, 23, 24]. Direct solution of the atomic eigenstate problem would make it possible to study *radiative* properties of the atom, too. This paper is devoted to an analysis of dynamics of a charged fermion in an external electromagnetic field within extended electrodynamics with a background axial vector b_μ. Approximate methods are used to implicitly solve the eigenstate problem in a central electric field; for the

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Coulomb field, an explicit solution is obtained (see section V). The quasi-relativistic approach is also employed to obtain the corrections the conventional Schrödinger-Pauli-Dirac equation acquires in the background of b_0 (sections III, IV). Moreover, in view of recent publications [25, 26, 27], some other types of couplings and the corresponding fermion eigenstates are also discussed. As an example of applying these results, an effect of b_0-induced asymmetry of the angular distribution of spontaneous radiation of a polarized hydrogen atom is demonstrated (section VI).

In addition, a polarized hydrogen atom is shown to have a nonzero *anapole* moment originally introduced in [18]. This characteristic is specific for parity-nonconserving systems interacting with electromagnetic field. For example, due to weak interaction loops, neutrinos can possess such a moment, and it is the only electromagnetic characteristic that maintains for Majorana neutrinos [28].

II. THE MODEL

We will restrict our consideration to a specific case of extended electrodynamics of electrons and photons within the SME (further refereed to as extended QED) with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(\frac{i}{2}\gamma^\mu \overleftrightarrow{D}_\mu - m_e - b_\mu\gamma^\mu\gamma_5\right)\psi, \tag{2.1}$$

$$D_\mu \equiv \partial_\mu + ieA_\mu(x), \qquad \gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.2}$$

where e, m_e are the electron charge and mass, respectively; $x^\mu \equiv \{ct, \boldsymbol{r}\}$, and b_μ is a constant CPT-odd axial vector. Present constraints on b_μ *for electron* are the following [16, 17, 29]:

$$|\, b_0 \,| \lesssim 10^{-2} \text{ eV}, \tag{2.3}$$

$$|\, \boldsymbol{b} \,| \lesssim 10^{-19} \text{ eV}, \tag{2.4}$$

while the constraints for *nucleons* are some orders more stringent. We will use one-particle approximation in the framework of relativistic quantum mechanics. Recalling (2.1), one can obtain the Hamiltonian for an electron in an external electromagnetic field

$$\hat{H}(t) = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{P}} + \beta m_e + eA_0(\hat{\boldsymbol{r}}, t) + b_0\gamma_5 + \boldsymbol{b}\cdot\boldsymbol{\Sigma}, \tag{2.5}$$

where $\hat{\boldsymbol{P}} \equiv \hat{\boldsymbol{p}} - e\boldsymbol{A}(\hat{\boldsymbol{r}}, t)$ and $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$, $\beta \equiv \gamma^0$, and $\boldsymbol{\Sigma} \equiv -\boldsymbol{\alpha}\gamma_5$. In Dirac spinor representation, spatial parity operator takes the form:

$$\hat{P}\xi(\boldsymbol{r}, t) \equiv \gamma^0\xi(-\boldsymbol{r}, t), \tag{2.6}$$

hence $\hat{P}^\dagger = \hat{P}$, $\hat{P}^\dagger\hat{P} = \hat{P}^2 = 1$. Hamiltonian (2.5) commutes with $\hat{P}$ if $A_0(\boldsymbol{r}, t) = A_0(-\boldsymbol{r}, t)$, $\boldsymbol{A}(\boldsymbol{r}, t) = -\boldsymbol{A}(-\boldsymbol{r}, t)$, and $b_0 = 0$. In particular, the presence of b_0 can violate the P-parity of Hamiltonian (2.5) in a spherically-symmetric field $A^\mu = \{\phi(r), \boldsymbol{0}\}$, in the Coulomb field of an infinitely heavy nucleus for example, with

$$\phi(r) = -\frac{Ze}{4\pi r}. \tag{2.7}$$

Charge conjugation of Hamiltonian (2.5) only changes the sign of electric charge e to the opposite; so hydrogen and anti-hydrogen atoms possess equivalent dynamics even if $b_\mu \neq 0$. Due to these facts our primary interest concerns the investigation of unusual properties of a hydrogen atom induced by the presence of a nonzero constant b_0.

III. $1/c^2$-APPROXIMATION FOR THE DIRAC EQUATION IN THE $b_\mu \neq 0$ CASE

The quasi-relativistic approximation assumes an expansion into a series with respect to $1/c$. It is thus necessary in this section to rewrite the Hamiltonian (2.5) in the CGS system of units with the speed of light $c \neq 1$

$$\hat{H} = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{P}} + \beta m_e c^2 + eA_0 + cb_t\gamma_5 + \boldsymbol{b}\cdot\boldsymbol{\Sigma}, \tag{3.1}$$

where $\hat{\boldsymbol{P}} \equiv \hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}(\hat{\boldsymbol{r}}, t)$ and $b_t \equiv b_0/c$ so that b_t has dimensionality of momentum. Consider the extended Dirac equation with the Hamiltonian (3.1) in a non-stationary external field $A_\mu(x)$:

$$i\hbar\frac{\partial\psi(\boldsymbol{r}, t)}{\partial t} = \hat{H}(t)\psi(\boldsymbol{r}, t), \tag{3.2}$$

$$\int \psi^\dagger(\boldsymbol{r}, t)\psi(\boldsymbol{r}, t)d^3r = 1. \tag{3.3}$$

Following the standard method (see, e.g. [30, 31]), let us shift the energy by means of a unitary transformation

$$\psi = \exp\left\{-i\frac{m_e c^2}{\hbar}t\right\}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.4}$$

In terms of 2-component spinors u and v, the Dirac equation (3.2) takes the form:

$$\begin{pmatrix} \hat{\lambda} & c\hat{\Lambda} \\ c\hat{\Lambda} & \hat{\lambda} - 2m_e c^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{3.5}$$

$$\hat{\Lambda} \equiv \boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}} - b_t, \quad \hat{\lambda} \equiv eA_0 + \boldsymbol{\sigma}\cdot\boldsymbol{b} - i\hbar\frac{\partial}{\partial t}, \tag{3.6}$$

where $\boldsymbol{\sigma}$ denotes a vector of the three Pauli matrices, and the Dirac matrices are taken in the standard representation:

$$\beta \equiv \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}. \tag{3.7}$$

Consider an electron in a state with a positive sign of energy (this in fact does not imply that the electron possesses a *definite energy*). External fields are assumed to be weak enough and to have frequencies much smaller than $m_e c^2/\hbar$ so that $\boldsymbol{E}, \boldsymbol{H}, \hat{\boldsymbol{P}}$, $i\hbar\frac{\partial}{\partial t} - eA_0 = O(c^0)$, when acting upon u, v.

In this section, we also assume that

$$b_t \equiv b_0/c = O(c^0), \qquad \boldsymbol{b} = O(c^0). \tag{3.8}$$

Contrary to the conventional electrodynamics, within the context of which the Gaussian units are usually used, the order of b_μ in $1/c$ is quite ambiguous. In electrodynamics, certain powers of $1/c$ can be assigned to the fields $\boldsymbol{E}, \boldsymbol{H}, A_\mu$ which result in a hierarchy of electromagnetic effects having different orders in $1/c$. For instance, radiative processes are at least of the third order in $1/c$, hence the $1/c^2$-approximation is worth considering. In contrast, the physical origin of b_μ is not yet finally established, and hence, we use the convention (3.8), due to the symmetry between $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}}$ and b_t, both entering (3.5). Together with (3.6), this convention implies

$$\hat{\lambda}, \hat{\Lambda} = O(c^0), \tag{3.9}$$

when acting upon u, v. Then the second line of (3.5) gives:

$$v = \frac{1}{2m_e c}\left(1 + \frac{\hat{\lambda}}{2m_e c^2}\right)\hat{\Lambda}u + O(1/c^4), \tag{3.10}$$

and v is thus suppressed, compared with u, for nonrelativistic positive-energy solutions. On the other hand, the square of the norm

$$\|\psi\|^2 \equiv \int \psi^\dagger\psi d^3r = 1 = O(c^0), \tag{3.11}$$

$$\|\psi\|^2 = \int (u^\dagger u + v^\dagger v)d^3r = \int u^\dagger(1 + O(1/c^2))u d^3r, \tag{3.12}$$

consequently, $u = O(c^0)$ and, due to (3.10), $v = O(1/c)$. This, in addition, results in suppression of the terms in the quasi-relativistic Hamiltonian stemming from the block-off-diagonal part of the matrix in (3.5), in particular, the terms containing b_0. As a result, the quasi-relativistic Hamiltonian will contain leading-order b_0-induced contributions proportional to $b_t = b_0/c$, but not b_0 itself (see (3.21)).

Now, instead of u, the following 2-component spinor field $\Phi(x) \in \mathbb{C}^2$ should be introduced as the quasi-relativistic wavefunction of the electron:

$$\Phi(x) \equiv \left(1 + \frac{\hat{\Lambda}^2}{8m_e^2 c^2}\right)u. \tag{3.13}$$

In this case, integration by parts shows the time evolution to preserve the norm

$$\begin{aligned}\|\Phi\|^2 \equiv \int \Phi^\dagger\Phi d^3r &= \int d^3r \left[\left(1+\frac{\hat\Lambda^2}{8m_e^2c^2}\right)u\right]^\dagger\left[\left(1+\frac{\hat\Lambda^2}{8m_e^2c^2}\right)u\right] = \\ &= \int d^3r\; u^\dagger\left(1+\frac{\hat\Lambda^2}{8m_e^2c^2}\right)^2 u = \int d^3r\; u^\dagger\left(1+\frac{\hat\Lambda^2}{4m_e^2c^2}\right)u + O(1/c^3) = \\ &= \int d^3r\left\{u^\dagger u + \left(\frac{\hat\Lambda u}{2m_ec}\right)^\dagger\left(\frac{\hat\Lambda u}{2m_ec}\right)\right\} + O(1/c^3) = \\ &= \int d^3r\;\left\{u^\dagger u + v^\dagger v\right\} + O(1/c^3) = \|\psi\|^2 + O(1/c^3) = 1 + O(1/c^3), \end{aligned} \tag{3.14}$$

while $\int u^\dagger u d^3r$ varies with time by a $O(1/c^2)$-amount. However, the transformation (3.13) leaves "probability distribution" $\Phi^\dagger\Phi$ different from $\psi^\dagger\psi$ by a fully-divergent term of the order $1/c^2$:

$$\Phi^\dagger\Phi = \psi^\dagger\psi + \operatorname{div}\mathbf{j}_{\rm ZB} + O(1/c^3), \tag{3.15}$$

$$(\mathrm{j}_{\rm ZB})_i = -\frac{1}{8m_e^2c^2}\left(\hbar^2\nabla_i(u^\dagger u) - 2\hbar\epsilon_{ijk}u^\dagger\sigma_j\hat P_k u\right). \tag{3.16}$$

This situation reflects the presence of negative-energy states resulting in the *Zitterbewegung* of the electron.

Now expressing u and v in terms of Φ using (3.10) and (3.13), write the first line of (3.5):

$$0 = \hat\lambda u + c\hat\Lambda v = \left\{\hat\lambda + \frac{1}{2m_e}\hat\Lambda\left(1+\frac{\hat\lambda}{2m_ec^2}\right)\hat\Lambda\right\}\left(1-\frac{\hat\Lambda^2}{8m_e^2c^2}\right)\Phi + O(1/c^3). \tag{3.17}$$

To obtain an equation in the form $i\hbar\,\partial\Phi/\partial t = \hat h\Phi$, one must make iterations to leave only one time derivative of Φ in the right side of (3.17). The corresponding operator is implicitly contained in $\hat\lambda$. First consider the above equation in the $1/c$-approximation:

$$\hat\lambda\Phi = -\frac{\hat\Lambda^2}{2m_e}\Phi + O(1/c^2). \tag{3.18}$$

After some transformations with the use of (3.18), we obtain in the $1/c^2$-approximation:

$$\left\{\hat\lambda + \frac{\hat\Lambda^2}{2m_e}\left(1-\frac{\hat\Lambda^2}{4m_e^2c^2}\right) - \frac{1}{8m_e^2c^2}\left[\left[\hat\lambda,\hat\Lambda\right],\hat\Lambda\right]\right\}\Phi = O(1/c^3). \tag{3.19}$$

Note that commutator $\left[\hat\lambda,\hat\Lambda\right]$ does not contain $\partial/\partial t$ operator, so there is *only one* time derivative of Φ in (3.19), namely the one contained in $\hat\lambda\Phi$. Converted into its usual form, (3.19) gives the quasi-relativistic equation for a positive-energy electron

$$i\hbar\frac{\partial\Phi}{\partial t} = \hat h\Phi, \qquad \psi^\dagger\psi = \Phi^\dagger\Phi - \operatorname{div}\mathbf{j}_{\rm ZB} + O(1/c^3); \tag{3.20}$$

$$\begin{aligned}\hat h &= \frac{\hat\Pi'^2}{2m_e}\left(1-\frac{\hat\Pi'^2}{4m_e^2c^2}\right) - \frac{e\hbar}{2m_ec}\boldsymbol{\sigma H} + \boldsymbol{\sigma b} + eA_0 - \\ &- \frac{e\hbar}{4m_e^2c^2}\boldsymbol{\sigma}[\boldsymbol{E}\hat{\boldsymbol{P}}] - \frac{e\hbar^2}{8m_e^2c^2}\operatorname{div}\boldsymbol{E} - \frac{\boldsymbol{\sigma}[\hat{\boldsymbol{P}}[\boldsymbol{b}\hat{\boldsymbol{P}}]]}{2m_e^2c^2}, \end{aligned} \tag{3.21}$$

$$\hat{\boldsymbol{\Pi}} \equiv \hat{\boldsymbol{P}} - b_t\boldsymbol{\sigma}, \tag{3.22}$$

$$\hat\Pi'^2 \equiv \hat{\boldsymbol{\Pi}}^2 - 2b_t^2 \equiv \hat{\boldsymbol{P}}^2 + b_t^2 - 2b_t\boldsymbol{\sigma}\cdot\hat{\boldsymbol{P}}. \tag{3.23}$$

The Hamiltonian $\hat h$ is precisely hermitian and the corresponding equations of motion demonstrate their exact local gauge invariance:

$$\hat h^\dagger[A_\mu] = \hat h[A_\mu], \tag{3.24}$$

$$\left(\hat h[A_\mu] - i\hbar\frac{\partial}{\partial t}\right)\exp\left\{i\frac{e}{\hbar c}\alpha(x)\right\} = \exp\left\{i\frac{e}{\hbar c}\alpha(x)\right\}\left(\hat h[A_\mu+\partial_\mu\alpha] - i\hbar\frac{\partial}{\partial t}\right) \qquad \forall\alpha(x)\in\mathbb{R}. \tag{3.25}$$

In the $1/c$-approximation, we arrive at the Pauli equation, through which the expressions for the probability current and density are easily found:

$$i\hbar\frac{\partial\Phi_P}{\partial t} = \hat{h}_P\Phi_P, \tag{3.26}$$

$$\hat{h}_P = \frac{\mathbf{\Pi}^2}{2m_e} - \frac{b_t^2}{m_e} - \frac{e\hbar}{2m_ec}\boldsymbol{\sigma}\boldsymbol{H} + eA_0 + \boldsymbol{\sigma}\boldsymbol{b}, \tag{3.27}$$

$$j_P^\mu = \left\{c\Phi_P^\dagger\Phi_P, \quad \frac{1}{2m_e}\left(\Phi_P^\dagger(\hat{\boldsymbol{P}}\Phi_P) + (\hat{\boldsymbol{P}}\Phi_P)^\dagger\Phi_P\right) - \frac{b_0}{m_ec}\Phi_P^\dagger\boldsymbol{\sigma}\Phi_P\right\}, \tag{3.28}$$

i.e. the current acquires an additional spin-dependent term in the $b_0 \neq 0$ case. The terms in (3.27) involving external fields form the interaction Hamiltonian. For $A_\mu(x)$ taken in the Coulomb gauge, it reads as follows:

$$\hat{h}_{P\,\mathrm{int}} = -\frac{e}{m_ec}\boldsymbol{A}\cdot\hat{\boldsymbol{\pi}} + eA_0 - \frac{e\hbar}{2m_ec}\boldsymbol{\sigma}\cdot\boldsymbol{H} + \frac{e^2}{2m_ec^2}\boldsymbol{A}^2, \tag{3.29}$$

$$\hat{\boldsymbol{\pi}} \equiv \hat{\boldsymbol{p}} - b_t\boldsymbol{\sigma}. \tag{3.30}$$

The difference of (3.29) from that in the conventional QED is generated by a gauge-like shift in the momentum space ($\hat{\boldsymbol{p}} \to \hat{\boldsymbol{\pi}}$). This feature will be used in section IV for constructing the solutions of the eigenstate problem.

The results obtained agree with those published in [25] and [26], in the corresponding particular cases. In the former paper, a nonrelativistic Hamiltonian for a *free* electron was obtained using the Foldy-Wouthysen method ($1/m_e$-series), within the first order approximation with respect to *all possible* SME-corrections in the fermion sector of extended QED [1, 2]. In the special case of the axial vector background b_μ, the resulting nonrelativistic Hamiltonian can be obtained from the formulas of paper [25]:

$$\hat{h}_{\mathrm{FW}} = \frac{\hat{\boldsymbol{p}}^2}{2m_e} + \boldsymbol{\sigma}\boldsymbol{b} - \frac{b_0}{m_ec}\boldsymbol{\sigma}\hat{\boldsymbol{p}} + \frac{\hat{p}_j\sigma_l}{2m_e^2c^2}(b_j\hat{p}_l - b_l\hat{p}_j) + \frac{b_0}{2m_e^3c^3}\hat{\boldsymbol{p}}^2(\boldsymbol{\sigma}\hat{\boldsymbol{p}}). \tag{3.31}$$

On the other hand, for a free electron, the Hamiltonian (3.21) takes the form:

$$\hat{h} = \frac{\hat{\boldsymbol{p}}^2 - 2b_t\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} + b_t^2}{2m_e}\left(1 - \frac{\hat{\boldsymbol{p}}^2 - 2b_t\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} + b_t^2}{4m_e^2c^2}\right) + \boldsymbol{\sigma}\boldsymbol{b} - \frac{\boldsymbol{\sigma}[\hat{\boldsymbol{p}}[\boldsymbol{b}\hat{\boldsymbol{p}}]]}{2m_e^2c^2}. \tag{3.32}$$

One can easily find that, within the linear order in b_μ and the third order in p/m_e (the approximation used in [25]), the two expressions are identical. The absence of the term proportional to p^4/m_e^3 in the former expression does not indicate an error. Instead, it is a consequence of the difference in the expansion parameters chosen, i.e. p/m_e and $1/c$, respectively.

It should be emphasized that the method used in paper [25] to obtain expression (3.31) was based on a series expansion of a *precise relativistic* Hamiltonian *for a 2-component wavefunction of a free particle* constructed using the Foldy-Wouthysen iterations [32]. Making these iterations, however, is inconvenient in the presence of external fields. In contrast, the method used in our paper takes these fields into account from the beginning.

Quasi-relativistic methods similar to those used in our paper were employed in [26] to find the $1/c$-corrections to the Dirac equation *in an external electromagnetic field* with additional a_μ and b_μ SME-couplings. In addition, plane wave solutions were obtained, and SME-specific modifications of the hydrogen spectrum were estimated, within the nonrelativistic approximation. For the b_μ coupling, the calculations performed have led to the Pauli Hamiltonian of the form (3.27). The contributions in the fermion Lagrangian and Hamiltonian, corresponding to a_μ coupling, are as follows [1, 2]:

$$\Delta\mathcal{L}^{(a)} = -\bar{\psi}\gamma^\mu a_\mu\psi, \tag{3.33}$$

$$\Delta\hat{H}^{(a)} = \gamma^0\gamma^\mu a_\mu = a_0 - \boldsymbol{\alpha}\cdot\boldsymbol{a}, \tag{3.34}$$

where a_μ is a constant background 4-vector, which can be treated as a vacuum expectation value of some Planck-scale fundamental fields. As mentioned in [1, 26], transition from the $a_\mu = 0$ to the $a_\mu \neq 0$ case is a kind of a gauge transformation because

$$A_\mu(x) \to A_\mu^{(a)}(x) = A_\mu(x) + \frac{1}{e}a_\mu = A_\mu(x) - \partial_\mu\alpha(x), \tag{3.35}$$

$$\alpha(x) = -\frac{1}{e}a_\mu x^\mu. \tag{3.36}$$

This feature makes it possible to find a system of exact solutions of the Dirac equation modified with the a_μ-term making an inverse gauge (and unitary) transformation. Suppose the eigenstate problem is solved in the $a_\mu = 0$ case so that

$$\hat{H}^{(0)}\psi_n^{(0)} = E_n^{(0)}\psi_n^{(0)}, \tag{3.37}$$

$$\hat{n}_i^{(0)}\psi_n^{(0)} = n_i\psi_n^{(0)}, \quad i = 1, 2, \ldots, N; \tag{3.38}$$

$$\left(\psi_m^{(0)}, \psi_n^{(0)}\right) \equiv \int d^3r\ \psi_m^{(0)\dagger}(\boldsymbol{r})\psi_n^{(0)}(\boldsymbol{r}) = \delta_{m,n}, \tag{3.39}$$

where $m = \{m_i\} \equiv \{m_1, m_2, \ldots, m_N\}$, $n = \{n_i\} \equiv \{n_1, n_2, \ldots, n_N\}$ denote the sets of quantum numbers corresponding to N hermitian operators $\hat{n}_i^{(0)}$ that should commute with $\hat{H}^{(0)}$ and with each other. The operators $\hat{n}_i^{(0)}$ are needed only to represent quantum numbers, i.e. they form a *complete set of observables*. It should be pointed out that the choice of these operators does not affect the eigenstate problem itself, but only forms the basis of the eigenstates and enumerates them. For example, for the nonrelativistic hydrogen atom the quantum numbers are usually taken such that $n_1 \equiv n$, $n_2 \equiv l$, $n_3 \equiv m$ define the eigenvalues of the three operators, namely $\hat{n}_1^{(0)} \equiv \hat{H}^{(0)}$, $\hat{n}_2^{(0)} \equiv \hat{\boldsymbol{l}}^2$, and $\hat{n}_3^{(0)} \equiv \hat{l}_3$, in the eigenstate $\psi_{nlm}^{(0)}$.

The system of solutions for $a_\mu \neq 0$ reads

$$\hat{H}^{(a)}\psi_n^{(a)}(\boldsymbol{r}) = E_n^{(a)}\psi_n^{(a)}(\boldsymbol{r}), \tag{3.40}$$

$$\hat{n}_i^{(a)}\psi_n^{(a)} = n_i\psi_n^{(a)}, \tag{3.41}$$

$$\left(\psi_m^{(a)}, \psi_n^{(a)}\right) = \delta_{m,n}; \tag{3.42}$$

$$\hat{H}^{(a)} = c\boldsymbol{\alpha}\left(\hat{\boldsymbol{p}} - \frac{1}{c}(e\boldsymbol{A} + \boldsymbol{a})\right) + m_e\beta + (eA_0 + a_0); \tag{3.43}$$

$$\psi_n^{(a)} = e^{i\boldsymbol{a}\cdot\boldsymbol{r}/\hbar c}\psi_n^{(0)}, \tag{3.44}$$

$$E_n^{(a)} = E_n^{(0)} + a_0. \tag{3.45}$$

The energy spectrum is shifted by a constant value a_0; no spectroscopic signature is therefore left by the presence of the nonzero a_μ (i.e. transition frequencies are unaffected). However, the meaning of the quantum numbers n_i (which run through the same set of values as in (3.38)) is changed, because, for $\psi_n^{(a)}$ functions, they correspond to the operators $\hat{n}_i^{(a)} \neq \hat{n}_i^{(0)}$ which can be readily constructed from $\hat{n}_i^{(0)}$:

$$\hat{n}_i^{(a)} = e^{i\boldsymbol{a}\hat{\boldsymbol{r}}/\hbar c}\hat{n}_i^{(0)}e^{-i\boldsymbol{a}\hat{\boldsymbol{r}}/\hbar c}. \tag{3.46}$$

For instance, in the case of a hydrogen atom, when the unitary transformation is made, $\psi_{nlm}^{(0)} \to \psi_{nlm}^{(a)}$, with a being some parameter of the transformation, the resulting $\psi_{nlm}^{(a)}$ is an eigenstate of the transformed Hamiltonian $H^{(a)}$, with the same eigenvalue (energy), but now quantum numbers n, l, m correspond to *new* operators $\hat{n}_i^{(a)} \neq \hat{n}_i^{(0)}$, $i = 1, 2, 3$.

Using (3.13) and (3.4), one can find that the transformation (3.44) maintains its form for the Pauli wavefunction Φ_P:

$$\Phi_{P,n}^{(a)}(\boldsymbol{r}) \ = \ e^{i\boldsymbol{a}\cdot\boldsymbol{r}}\Phi_{P,n}(\boldsymbol{r}), \tag{3.47}$$

while the nonrelativistic spectrum and $\hat{n}_i$ operators are still transformed following (3.45) and (3.46), respectively.

Another investigation, which is worth mentioning, was held in [27]. The authors have considered two non-minimal Lorentz-violating couplings in the fermion sector of QED:

$$\Delta\mathcal{L}^{(g,g_{\rm a})} = \bar{\psi}(-gv^\nu + g_{\rm a}v_{\rm a}^\nu\gamma_5)\gamma^\mu F_{\mu\nu}^*\psi, \tag{3.48}$$

where g and $g_{\rm a}$ are the coupling constants while v^ν and $v_{\rm a}^\nu$ are fixed background vectors ('a' is not a component index but means 'axial'), and $F_{\mu\nu}^* = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual field tensor. We leave aside the question of the origin of such couplings; for more information, the reader is referred to [27] and the references therein. The authors of the paper also used the nonrelativistic Pauli approach to obtain the Pauli equation in the presence of the background vectors v^ν and $v_{\rm a}^\nu$, and then calculated the first-order energy corrections using perturbation theory. In addition, they considered an atom in a homogeneous external magnetic field also treated perturbatively. The case of the g and $g_{\rm a}$ couplings, in general, does not permit such an easy construction of the eigenfunctions as the case of the a_μ coupling. However,

in one special case not considered in [27], namely for the constant homogeneous external field $F_{\mu\nu}(x) = const$ and $g_{\rm a} = 0$, the eigenstate problem can be solved in much the same way as described in (3.44) and (3.45). Indeed, the transformation analogous to (3.44) reads as follows:

$$\psi_n^{(g)}(\boldsymbol{r}) \;=\; \exp\left\{-\frac{ig}{\hbar c}\boldsymbol{r}\left([\boldsymbol{vE}]-v^0\boldsymbol{H}\right)\right\}\psi_n^{(0)}(\boldsymbol{r}), \tag{3.49}$$

$$E_n^{(g)} \;=\; E_n^{(0)} + \boldsymbol{v}\cdot\boldsymbol{H}. \tag{3.50}$$

Again, the spectrum is shifted by a constant value, though depending on the direction of the magnetic field. However, the change in the wavefunctions could possibly affect, for instance, the properties of synchrotron radiation in a homogeneous magnetic field.

Nonetheless, we confine ourselves to demonstrating the prospects of unitary transformations for solving wave equations containing Lorentz-violating terms. In the following sections, similar techniques will be used to obtain the solutions in the case of the b_μ coupling, and to study the dynamics of a bound electron in such a background.

IV. HYDROGEN-LIKE ATOM. QUASI-RELATIVISTIC APPROACH

Consider first the Pauli Hamiltonian (3.27) within the first order in b_μ:

$$\hat{h}_P = \frac{\hat{\boldsymbol{\Pi}}^2}{2m_e} + eA_0 - \frac{e\hbar}{2m_e c}\boldsymbol{\sigma H} + \boldsymbol{\sigma b}. \tag{4.1}$$

We suppose that $A_\mu(x)$ is taken in the Coulomb gauge with

$$\frac{\partial A_0}{\partial t} \;=\; 0, \tag{4.2}$$

$$\mathrm{div}\,\boldsymbol{A} \;=\; 0. \tag{4.3}$$

Make an inverse gauge-like shift of the momentum ($\hat{\boldsymbol{\Pi}} \to \hat{\boldsymbol{P}}$) performing a *unitary* transformation:

$$\Phi_P \;\to\; \Phi'_P = \hat{U}_P\Phi_P, \quad \hat{h}_P \to \hat{h}'_P = \hat{U}_P\hat{h}_P\hat{U}_P^\dagger, \qquad \hat{U}_P \equiv \exp\left\{-\frac{ib_t}{\hbar}\boldsymbol{\sigma}\cdot\boldsymbol{r}\right\}; \tag{4.4}$$

$$\hat{h}'_P \;=\; \frac{\hat{\boldsymbol{P}}^2}{2m_e} + eA_0 - \left(\frac{e\hbar}{2m_e c}\boldsymbol{\sigma} + \hat{\boldsymbol{\mu}}_A\right)\boldsymbol{H} + \boldsymbol{\sigma}\cdot\boldsymbol{b}, \tag{4.5}$$

$$\hat{\boldsymbol{\mu}}_A \;=\; \frac{eb_t}{m_e c}[\boldsymbol{\sigma r}]. \tag{4.6}$$

It is clear that the transformation reduces the Lorentz-violating interaction to a modification of the electron magnetic moment, which acquires a CPT-odd correction $\hat{\boldsymbol{\mu}}_A$. Consequently, the terms of the first order in b_t vanish in the transformed Hamiltonian as the external magnetic field $\boldsymbol{H}$ is turned off. In particular, within the approximation used, the eigenstate problem in an *electric* field would look quite conventional after the transformation. In a relativistic theory discussed in section V, an *electric* dipole moment correction also arises but it vanishes in the nonrelativistic approximation.

Let $A^\mu = \{\phi(r), \boldsymbol{0}\}$ and $\boldsymbol{b} = \{0, 0, b_z\}$, $\phi(r)$ being the potential of the nucleus initially considered as spherically-symmetric, but not mandatory the Coulomb potential.[1] The problem resembles that of an electron in a homogeneous magnetic field $\boldsymbol{H}_b$ but for the only difference: now there is no coupling in the kinetic term (that is, we have $\hat{\boldsymbol{p}}$ for the momentum instead of $\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}_b$, where $\boldsymbol{H}_b = \mathrm{rot}\,\boldsymbol{A}_b$). The coupling to the external field $\boldsymbol{b}$ involves only the spin degrees of freedom but not the orbital ones. The energy eigenstates can be easily obtained in the transformed

[1] Indeed, due to radiative corrections, Coulomb attraction becomes stronger than $\sim 1/r$ at short distances contributing in the Lamb shift of electron eigenstates [34]. The Lamb shift however originates from the three one-loop corrections to electrodynamics including electron mass renormalization, its anomalous magnetic moment and the modification of the Coulomb law. For s-states, the first of them makes a major contribution to the Lamb shift.

representation and then transformed back to the initial one:

$$(\Phi'_P)_{nlm_lm_s}(\boldsymbol{r}) = R_{nl}(r)Y_{l,m_l}(\boldsymbol{r}/r)\chi_{m_s}; \tag{4.7}$$

$$(\Phi_P)_{nlm_lm_s}(\boldsymbol{r}) = R_{nl}(r)Y_{l,m_l}(\boldsymbol{r}/r)\left(1+\frac{ib_t}{\hbar}\boldsymbol{\sigma}\cdot\boldsymbol{r}\right)\chi_{m_s}; \tag{4.8}$$

$$E_{nlm_s} = E^{(0)}_{nl} + 2b_z m_s, \tag{4.9}$$

where $n=1,2,3,\ldots$, $l=\overline{0,n-1}$, $m_l=\overline{-l,l}$, $m_s=\pm 1/2$ are the quantum numbers denoted according to a common convention; χ_{m_s} are the spin z-component eigenvectors. $R_{nl}(r)$ and $E^{(0)}_{nl}$ are the radial wavefunction and the energy in the $b_0=0$ case, respectively. In the Coulomb case, we have [33]

$$R_{nl}(r) = \frac{2Z^{3/2}}{n^2 r_{\mathrm{B}}^{3/2}}\sqrt{\frac{(n-l-1)!}{(n+l)!}}e^{-\rho/2}\rho^l L^{(2l+1)}_{n-l-1}(\rho), \tag{4.10}$$

$$E^{(0)}_{nl}\to E^{(0)}_n = -\frac{Z^2\hbar R}{n^2}, \tag{4.11}$$

where $\rho=2Zr/nr_{\mathrm{B}}$, $r_{\mathrm{B}}=\hbar^2/m_e e^2$ is the Bohr radius, $R=m_e e^4/2\hbar^3$ is the Rydberg constant, and $L^{(\nu)}_k$ denote the generalized Laguerre polynomials:

$$L^{(\nu)}_k(\rho)=\frac{1}{k!}\rho^{-\nu}e^{\rho}\frac{d^n}{d\rho^n}\left(\rho^{\nu+n}e^{-\rho}\right), \quad \mathtt{Re}\,\nu>0, \quad k=0,1,2,\ldots \tag{4.12}$$

The solution obtained shows that, with respect to the transformed representation, the only effect the presence of $\boldsymbol{b}$ generates *in the leading order* is a removed degeneracy over spin quantum number m_s, with the energy splitting being $\lesssim 10^{-4}$Hz. Neither the spectrum nor the eigenfunctions are affected by b_0, *only* the interaction with the external magnetic field is.

The $\boldsymbol{b}$-induced energy splitting into a doublet is a formal result of solving the eigenstate problem in a $1/c$-approximation that does not hold true when the spin-orbit interaction is considered that removes the degeneracy over quantum number j. The correct splitting magnitude can be estimated by means of a perturbation theory. In the absence of $\boldsymbol{b}$, the spectrum remains degenerate over l and m_j. The action of the perturbation term $\boldsymbol{\sigma}\cdot\boldsymbol{b}$, however, preserves these quantum numbers, so the perturbation theory can be applied to the atom as to a non-degenerate system. This situation is typical for the anomalous Zeeman effect [31, 33].

For the $\boldsymbol{\sigma b}$ term, the energy correction was first estimated in [26] but we shall do it once again. First, let $b^\mu=\{0,0,0,b_z\}$. Following the arguments explained in the preceding paragraph, take $|nljm_j\rangle$ for the eigenstates in the $\boldsymbol{b}=0$ case. Using the general expressions for them [30, 31],

$$\langle\boldsymbol{r}|nljm_j\rangle = R_{nlj}(r)Y^l_{jm_j}(\boldsymbol{r}/r), \tag{4.13}$$

$$Y^l_{jm_j} = \frac{1}{\sqrt{2l+1}}\begin{pmatrix}\sqrt{l+1/2+\varkappa m_j}\\ \varkappa\sqrt{l+1/2-\varkappa m_j}\end{pmatrix}, \tag{4.14}$$

$$\varkappa \equiv (-1)^{(l-l'+1)/2}=\pm 1 \text{ for } j=l\pm 1/2, \qquad l'\equiv 2j-l, \tag{4.15}$$

we obtain:

$$\Delta E^{(\boldsymbol{b})}_{nljm_j}=\langle nljm_j|\,\boldsymbol{\sigma b}\,|nljm_j\rangle = \int\limits_0^\infty R^2_{nlj}(r)r^2dr\cdot\frac{b_z}{2l+1}\left((l+1/2+\varkappa m_j)-(l+1/2-\varkappa m_j)\right)=$$

$$=\int\limits_0^\infty R^2_{nlj}(r)r^2dr\cdot\frac{2\varkappa m_j}{2l+1}b_z=\frac{2\varkappa m_j}{2l+1}b_z, \tag{4.16}$$

that is, twice the result obtained in [26]. Since the corrections induced by $\boldsymbol{b}$ are minuscule, we will further treat b^μ as a purely timelike 4-vector, with the time component b_0.

In search for b_0-corrections to the eigenstates, we shall resort to the $1/c^2$-approximation in the eigenstate problem. Consider the Coulomb case with $e\phi(r)=-Ze^2/r$ within the first-order approximation in $b^\mu=\{cb_t,\boldsymbol{0}\}$. A spectacular feature of this case is that the solutions can be explicitly expressed via their conventional form (for $b_0=0$). The

correspondence is generated again with a unitary transformation:

$$\hat{h} \;=\; \hat{U}^{\dagger}\hat{h}|_{b_0=0}\hat{U}, \qquad \hat{U}=\exp\left\{-\frac{ib_t}{\hbar}\left(1+\frac{Ze^2}{2m_ec^2r}\right)\boldsymbol{\sigma}\cdot\boldsymbol{r}\right\}, \tag{4.17}$$

$$\hat{h} \;=\; \frac{\hat{\boldsymbol{\pi}}^2}{2m_e}\left(1-\frac{\hat{\boldsymbol{\pi}}^2}{4m_e^2c^2}\right)-\frac{Ze^2}{r}+\frac{Ze^2\hbar^2}{4m_e^2c^2}\left(\frac{\boldsymbol{\sigma}\hat{\boldsymbol{l}}}{r^3}+2\pi\delta(\boldsymbol{r})\right). \tag{4.18}$$

As a result we obtain

$$\Phi_{nljm_j}(\boldsymbol{r}) \;=\; R_{nlj}(r)\left\{Y^l_{jm_j}(\boldsymbol{r}/r)-\frac{\varkappa b_t r}{\hbar}\left(1+\frac{Ze^2}{2m_ec^2r}\right)Y^{l'}_{jm_j}(\boldsymbol{r}/r)\right\}, \tag{4.19}$$

$$E \;=\; E^{(0)}_{nj}=-\frac{Z^2\hbar R}{n^2}\left[1+\frac{Z^2\alpha^2}{n}\left(\frac{1}{j+1/2}-\frac{3}{4n}\right)\right], \tag{4.20}$$

where $\varkappa$ and l' are defined in (4.15). The radial functions $R_{nlj}(r)$ remain the same as in the $b_0=0$ case (see (4.13)). In the nonrelativistic limit, they take the form (4.10).

Thus, no corrections to the energy spectrum are present due to b_0, within the $1/c^2$-approximation. Further analysis will show that there are no corrections of the first order in b_0 (see section V). Nevertheless, the perturbative method used in [26] to retrieve the energy corrections due to the term $-\frac{b_0}{m_ec}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}$ is incorrect. The spectrum is degenerate over l if $b_0=0$, while the perturbation operator is P-odd, and hence changes the l quantum number. The expectation value of such an operator clearly vanishes in a state possessing a definite l, and thus definite parity $P=(-1)^l$. The energy shift may not vanish, however, for some superposition of the states with opposite parities. This is common for the *linear Stark effect* [31, 33] that occurs due to a degeneracy of the hydrogen spectrum. The perturbation theory for a degenerate system must be employed instead of a simple averaging. Despite the above remarks, the methods employed in [26], have led to correct results.

In conclusion, we will show how the correction to the magnetic moment may cause an appearance of a nonzero *anapole* moment of the atomic *orbital* [18]. This is a classical quantity ascribed to a *parity-nonconserving* system (such systems exist in the conventional Standard Model due to weak interaction [28]) adding an interaction term of the form $-\boldsymbol{T}_{\rm Z}\cdot\mathrm{rot}\,\boldsymbol{H}$ to the Hamiltonian of the system, with $\boldsymbol{T}_{\rm Z}$ being the anapole moment. Consider a hydrogen atom in the ground state $1s_{1/2,m_j}$ where lower indices indicate the electron total angular momentum and its z-projection. Averaging the CPT-odd term $-\hat{\boldsymbol{\mu}}_A\cdot\boldsymbol{H}$ in this state yields:

$$V_{\rm Z}\equiv\langle-\hat{\boldsymbol{\mu}}_A\cdot\boldsymbol{H}(\hat{\boldsymbol{r}})\rangle=-\left\langle\hat{\boldsymbol{\mu}}_A\left(\boldsymbol{H}(\boldsymbol{0})+(\hat{\boldsymbol{r}}\cdot\boldsymbol{\nabla})\boldsymbol{H}(\boldsymbol{0})+\ldots\right)\right\rangle, \tag{4.21}$$

where $\boldsymbol{r}=\boldsymbol{0}$ points to the center of the Coulomb field. The ground $1s$ state possesses a definite parity $(-1)^l=+1$ and, in addition, a spherical symmetry, hence

$$\langle\hat{\boldsymbol{\mu}}_A\rangle \;=\; 0, \tag{4.22}$$

$$\langle\hat{x}_i\hat{x}_k\rangle \;=\; \frac{1}{3}\delta_{ik}\left\langle\hat{r}^2\right\rangle, \qquad i,k=1,2,3. \tag{4.23}$$

With the help of expressions (4.7) and (4.10), one can easily find that

$$\langle\sigma_i\hat{x}_k\hat{x}_n\rangle=2r_{\rm B}^2\delta_{i3}\delta_{kn}\cdot m_j, \qquad i,k,n=1,2,3. \tag{4.24}$$

which results in the following:

$$V_{\rm Z} \;\approx\; -\frac{2eb_0r_{\rm B}^2}{m_ec^2}m_j\epsilon_{3ik}\partial_iH_k=-\boldsymbol{T}_{\rm Z}\cdot\mathrm{rot}\,\boldsymbol{H}, \tag{4.25}$$

$$\boldsymbol{T}_{\rm Z} \;=\; 2er_{\rm B}^2\left(\frac{b_0}{m_ec^2}\right)m_j\boldsymbol{e}_3, \tag{4.26}$$

where $\boldsymbol{e}_3$ is the basis unit vector along the z-axis.

V. SERIES EXPANSION OF THE DIRAC EQUATION WITH RESPECT TO b_0

In this section we discuss the case $b^\mu=\{b^0,\boldsymbol{0}\}$ and $A^\mu=\{A^{(\rm e)}_0(x)+\phi(r),\boldsymbol{A}^{(\rm e)}(x)\}$ using the Heaviside units, with $\hbar=c=1$, $\alpha=\frac{e^2}{4\pi}$. Consider the Hamiltonian (2.5) and transform the corresponding wave equation using the

gauge-invariant unitary transformation:

$$\psi(x) \;\to\; \tilde{\psi}(x) = e^{-ib_0\hat{\Delta}_A}\psi(x), \tag{5.1}$$

$$\hat{H} - i\frac{\partial}{\partial t} \;\to\; \hat{\tilde{H}} - i\frac{\partial}{\partial t} = e^{-ib_0\hat{\Delta}_A}\left(\hat{H} - i\frac{\partial}{\partial t}\right)e^{ib_0\hat{\Delta}_A}; \tag{5.2}$$

$$\hat{\Delta}_A \;=\; \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{r}} - \frac{i}{m_e}(\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{L}}+1)\gamma^0\gamma_5, \tag{5.3}$$

$$\hat{\boldsymbol{L}} \;=\; [\hat{\boldsymbol{r}}\hat{\boldsymbol{P}}] = -[\hat{\boldsymbol{P}}\hat{\boldsymbol{r}}]. \tag{5.4}$$

Restricting ourselves to the second-order approximation in b_0, we obtain:

$$\hat{\tilde{H}} \approx \boldsymbol{\alpha}(\hat{\boldsymbol{p}} - e\boldsymbol{A}^{(\mathrm{e})}) + \beta m_e + e(\phi + A_0^{(\mathrm{e})}) - \frac{b_0^2}{m_e}\hat{f}\gamma^0 - \hat{\boldsymbol{d}}_A\boldsymbol{E}^{(\mathrm{e})} - \hat{\boldsymbol{\mu}}_A\boldsymbol{H}^{(\mathrm{e})} + H^{(2)}_{\mathrm{int}}[A^{(\mathrm{e})}], \tag{5.5}$$

with $\hat{f} \equiv \boldsymbol{\Sigma}\hat{\boldsymbol{l}} + 1$. $H^{(2)}_{\mathrm{int}}[A^{(\mathrm{e})}]$ stands for the second-order terms in b_0 describing the interaction with the external field $A^{(\mathrm{e})}_\mu$. Additional electric and magnetic dipole moment operators read as follows:

$$\hat{\boldsymbol{\mu}}_A \;=\; \frac{eb_0}{m_e}\gamma^0[\boldsymbol{\Sigma}\boldsymbol{r}], \tag{5.6}$$

$$\hat{\boldsymbol{d}}_A \;=\; -i\gamma_5\hat{\boldsymbol{\mu}}_A = -\frac{ieb_0}{m_e}[\boldsymbol{\gamma}\boldsymbol{r}]. \tag{5.7}$$

As we can see, no non-linear terms in the external field are present up to the first order in b_0, inclusively. Moreover, the moment $\hat{\boldsymbol{d}}_A$ couples with the *external* field only but *not* with the spherically-symmetric 'background' field $\phi(r)$, because for such a field $\hat{\boldsymbol{d}}_A\cdot(-\boldsymbol{\nabla}\phi) = 0$. The same situation holds in every higher order of the expansion due to the fact that $[\hat{\Delta}_A, \phi(r)] = 0$. For the same reason, the expressions for operators $\hat{\boldsymbol{d}}_A$ and $\hat{\boldsymbol{\mu}}_A$ are not affected by $\phi(r)$, in particular, they maintain their form for a *free* electron. Our approach however is applicable only to systems with the effective size much less than $1/b_0 \gtrsim 10^{-3}$cm.

We did not obtain a CPT-odd correction coupling to the electric field in the $1/c$-approximation because, in contrast to $\hat{\boldsymbol{\mu}}_A$, $\hat{\boldsymbol{d}}_A$ is a block-off-diagonal operator,

$$\hat{\boldsymbol{d}}_A = -\frac{ieb_t}{m_e c}\begin{pmatrix} 0 & [\boldsymbol{\sigma}\boldsymbol{x}] \\ -[\boldsymbol{\sigma}\boldsymbol{x}] & 0 \end{pmatrix}, \tag{5.8}$$

which mixes the 'upper' and the 'lower' 2-component spinors of the wavefunction. The 'lower' spinor vanishes in the non-relativistic limit (see eq. (3.10)), and so does the operator $\hat{\boldsymbol{d}}_A$. Instead, $\hat{\boldsymbol{\mu}}_A$ is a block-diagonal matrix that mixes the 'upper' spinors with themselves and consequently it does not vanish in the nonrelativistic limit.

Let $A^{(\mathrm{e})}_\mu = 0$, then the spherical symmetry allows us to search for the eigenfunctions in the form

$$\tilde{\psi}_{n_r l j m_j}(\boldsymbol{r}, t) = \begin{pmatrix} R^{(u)}_{n_r l j}(r) Y^l_{j m_j}(\boldsymbol{r}/r) \\ \varkappa R^{(v)}_{n_r l j}(r) Y^{l'}_{j m_j}(\boldsymbol{r}/r) \end{pmatrix}, \tag{5.9}$$

where $n_r \equiv n - j - 1/2$ is the radial quantum number and n is the principal quantum number, and l, as usual in the relativistic theory, determines the parity of the state $P = (-1)^l$, but not its orbital momentum. Operators $\hat{\boldsymbol{d}}_A$ and $\hat{\boldsymbol{\mu}}_A$ have vanishing expectation values in such a state. $\tilde{\psi}$ is the eigenfunction of the operator $\hat{f}\gamma^0$, with the eigenvalue $f \equiv \varkappa(j + 1/2)$. In the case under consideration, the transformed Hamiltonian (5.5) is the sum of its conventional value (for $b_0 = 0$) and a term proportional to $\hat{f}\gamma^0$; $\tilde{\psi}$ is an eigenfunction for both of them if the radial functions $R^{(u,v)}$ are taken the same as those for the $b_0 = 0$ case. The energy value which responds to $\tilde{\psi}$ is

$$E = \tilde{E} = E^{(0)}_{n_r l j} - \varkappa(j + 1/2)\frac{b_0^2}{m_e} = E^{(0)}_{n_r l j} \pm (j + 1/2)\frac{b_0^2}{m_e} \quad \text{for } l = j \pm 1/2. \tag{5.10}$$

An additional b_0-induced second-order energy splitting therefore arises:

$$\Delta E(j) \equiv E_{n_r, j+1/2,\, j} - E_{n_r, j-1/2,\, j} = (2j + 1)\frac{b_0^2}{m_e}. \tag{5.11}$$

This term originates from parity violation due to the b_μ-induced violation of CPT and removes the degeneracy over l in the Coulomb field case. $|\Delta E(j)| \lesssim 10^5$Hz for $j = 1/2$, that is, four orders of magnitude smaller than the Lamb shift [34]. Nonetheless, in contrast to the latter one existing mainly for s-states, the splitting (5.11) *increases* with growing j.

The eigenfunctions in the initial representation are obtained after performing the inverse transformation:

$$\psi_{n_r l j m_j}(\boldsymbol{r}) = e^{-b_0^2 f^2/2m_e^2} e^{-b_0^2 r^2/2} \begin{pmatrix} R^{(u)} Y^l_{jm_j} + b_0 \varkappa \left(\frac{f}{m_e} R^{(v)} - r R^{(u)} \right) Y^{l'}_{jm_j} \\ \varkappa R^{(v)} Y^{l'}_{jm_j} + b_0 \left(\frac{f}{m_e} R^{(u)} + r R^{(v)} \right) Y^l_{jm_j} \end{pmatrix}. \tag{5.12}$$

The presence of the admixture of spherical spinors with the different value of the orbital quantum number (l') breaks the parity of the states. 'Probability distribution' $\psi^\dagger\psi$ is not affected however, compared with the conventional ($b_0 = 0$) case, within the chosen approximation. Since the conventional solution in the Coulomb case is well-known [30], we can explicitly find the second-order approximation for the eigenfunctions in the $b_0 \neq 0$ Coulomb case:

$$E^{(0)}_{n_r j} = m_e \left(1 + \left(\frac{Z\alpha}{\gamma + n_r} \right)^2 \right)^{-1/2}, \tag{5.13}$$

$$\left. \begin{matrix} R^{(u)} \\ R^{(v)} \end{matrix} \right\} = \pm (2\lambda)^{3/2} \left(\frac{(m_e \pm E^{(0)}_{n_r j}) n_r!}{4 m_e \frac{Z\alpha m_e}{\lambda} \left(f + \frac{Z\alpha m_e}{\lambda} \right) \Gamma(2\gamma + n_r + 1)} \right)^{1/2} e^{-\lambda r} (2\lambda r)^{\gamma - 1} \times$$
$$\times \left(\left(f + \frac{Z\alpha m_e}{\lambda} \right) L^{(2\gamma)}_{n_r}(2\lambda r) \pm (1 - \delta_{n_r, 0})(2\gamma + n_r) L^{(2\gamma)}_{n_r - 1}(2\lambda r) \right), \tag{5.14}$$

$$\lambda \equiv \sqrt{m_e^2 - {E^{(0)}_{n_r j}}^2}, \qquad \gamma = \sqrt{(j + 1/2)^2 - (Z\alpha)^2}, \tag{5.15}$$

with $L^{(2\gamma)}_{n_r}$ being the generalized Laguerre polynomials defined in (4.12).The expression for the energy demonstrates an alternative mechanism of removing the degeneracy over l, different from that connected with the one-loop corrections in quantum electrodynamics.

VI. SPECIFIC RADIATIVE PROPERTIES OF A HYDROGEN ATOM INDUCED BY b_0

Finally, we demonstrate an example with CPT and Lorentz violation leading to radiative effects specific for the $b_0 \neq 0$ case and *linear* in b_0. Following the system of units convention used in section III, we assume $b^\mu = \{cb_t, \mathbf{0}\}$. Since the primary goal of this section is to obtain the leading-order b_0-induced terms in the radiation distribution, we restrict ourselves to the Pauli approximation and consider the radiation of a hydrogen ($Z = 1$) atom. Upon the transformation (4.5), the only term remaining with b_0 is $-\hat{\boldsymbol{\mu}}_A \cdot \boldsymbol{H}$. This term violates the spatial parity of the atom. With the use of the standard formulas [31], we find the angular distribution of spontaneous radiation probability:

$$\frac{dW_{fi}(\boldsymbol{k}, \lambda)}{d\Omega_{\boldsymbol{k}}} = \frac{\omega^3}{2\pi\hbar c^3} \left| \boldsymbol{e}^{(\lambda)*}(\boldsymbol{k}) \cdot \mathfrak{m}_{fi}(\boldsymbol{k}) \right|^2, \tag{6.1}$$

$$|\boldsymbol{k}| = \omega/c = (E_i - E_f)/\hbar c > 0, \qquad \lambda = 1, 2; \tag{6.2}$$

$$\hat{\mathfrak{m}} = e\hat{\boldsymbol{r}} - \frac{ie}{2}(\boldsymbol{k} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \left[\frac{\boldsymbol{k}}{k} \times \hat{\boldsymbol{\mu}} \right], \tag{6.3}$$

$$\hat{\boldsymbol{\mu}} = \frac{e\hbar}{2m_e c}(\hat{\boldsymbol{l}} + \boldsymbol{\sigma}) + \hat{\boldsymbol{\mu}}_A, \tag{6.4}$$

where $\boldsymbol{k}, \lambda$ are the photon momentum and polarization, and $\boldsymbol{e}^{(\lambda)}(\boldsymbol{k})$ is the polarization vector. $\langle f|$ and $|i\rangle$ denote the final and the initial electron states. The correction $\hat{\boldsymbol{\mu}}_A$ to the magnetic moment operator $\hat{\boldsymbol{\mu}}$ is defined in (4.6).

Radiation processes allowed due to the parity-violating interaction with $\hat{\boldsymbol{\mu}}_A$ (further referred to as Aj-radiation with j denoting the photon angular moment) are restricted by the same selection rules as those for the $E1$-radiation, the corresponding matrix element having the form typical for $M1$-radiation. Thus $A1$- and $E1$-photons have the same multipolity but the opposite parity. Linear in b_0 corrections to the angular distribution occur due to the interference between the $E1$- and the $A1$-radiation. Consequently, they vanish over the whole sphere because of the cancellation of spherical spinors with different parities, resulting in no linear in b_0 terms present in the *total* transition rate.

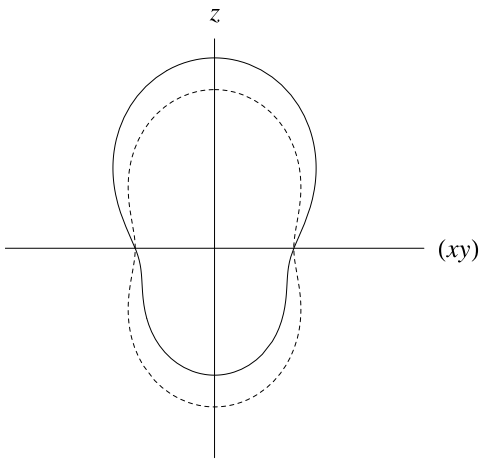

FIG. 1: Angular distribution of spontaneous radiation for $2p_{1/2,1/2} \to 1s_{1/2,-1/2}$ transition

The interference term does not vanish, for example, for the transition $2p_{1/2,1/2} \to 1s_{1/2,-1/2}$. The calculations show that after averaging over the photon polarizations $\lambda = 1, 2$, the resulting angular distribution reads (Θ is the angle between $\boldsymbol{k}$ and the z-axis):

$$\frac{dW}{d\Omega_{\boldsymbol{k}}} = \frac{512\alpha^3 R}{6561\pi}\left\{1 + \cos^2\Theta + \frac{8b_0}{m_e c^2}\cos\Theta\right\}. \tag{6.5}$$

As we can see, the presence of b_0 induces the violation of the conventional '$\boldsymbol{k}$-parity' of the distribution (the radiation rates in the opposite directions differ in the $b_0 \neq 0$ case). The relative magnitude of this violation is of the order $|b_0|/m_e c^2 \lesssim 2 \cdot 10^{-8}$. Distribution (6.5) is depicted in fig.1, with the dotted curve related to the $b_0 = 0$ case. To make the picture more vivid, we chose $b_0/m_e c^2 = 0.05$.

For unpolarized atoms, i.e. after averaging over m_j, $m_j{}'$ quantum numbers, the spherical symmetry is restored in the distribution, with no linear in b_0 $\boldsymbol{k}$-odd contributions present. This is the consequence of $SO(3)$-invariance unbroken even in the $b_0 \neq 0$ case (while $O(3)$ symmetry is broken since b_0 is a pseudoscalar).

We left aside the problem of polarization of atoms. If one uses Zeeman effect in a homogeneous magnetic field to obtain the polarization, then this magnetic field would also lead to parity violation due to the interaction with $\hat{\boldsymbol{\mu}}_A$. Another way the external magnetic field can break the atomic P-parity is that in the reference frame of a moving atom, an additional electric field will be induced that breaks the parity. The distribution of radiation of moving atoms can also be shifted due to aberration. These problems need further consideration. In the present paper however, we just demonstrated yet another scenario of P-parity violation in atomic transitions.

VII. CONCLUSION

In this paper, we have considered several solutions of the Dirac equation in the framework of the Standard Model Extension with particular types of Lorentz violation. The $1/c^2$-approximation for the extended Dirac equation was derived in the background of the axial vector SME-coupling b_μ. The expansion of the *relativistic* Dirac equation with respect to b_0 has been employed to solve the eigenstate problem for an electron in a spherically-symmetric potential well. The unitary transformation was found that was used to express the solutions with $b_0 \neq 0$ in terms of solutions for $b_0 = 0$, with the second order accuracy with respect to b_0. Explicit solutions have been obtained in the case of the Coulomb potential, demonstrating a specific b_0-quadratic energy splitting. The degeneracy over the orbital quantum number is removed, and it was shown that the corresponding energy splitting does not vanish for large j.

In addition, unitary transformations were used to obtain the *exact* eigenstates in the case of the coupling $-g\bar{\psi}\gamma^\mu v^\nu F^*_{\mu\nu}\psi$ with constant homogeneous electromagnetic field $F_{\mu\nu}$ [27].

The unitary transformation made it possible to obtain the b_0-corrections to the operators of $E1$ and $M1$ moments of the electron. These moments effectively lead to an existence of the *anapole* moment of the orbital [18].

Finally, the distribution of spontaneous radiation of a polarized hydrogen atom was shown to lose its central symmetry in the $b_0 \neq 0$ case, due to the violation of spatial parity. The results obtained can be treated only as an

illustration of the application of the model adopted. There are other physical effects that should also be considered together with the one discussed in this paper.

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